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Mesoscopic colonization of a spectral band

M Bertola^{1,2}, S Y Lee² and M Y Mo³

¹ Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve W, Montréal, Québec, Canada H3G 1M8

² Centre de recherches mathématiques, Université de Montréal, Canada

³ Department of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, UK

E-mail: bertola@crm.umontreal.ca

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Abstract

We consider the unitary matrix model in the limit where the size of the matrices becomes infinite and in the critical situation when a new spectral band is about to emerge. In previous works, the number of expected eigenvalues in the neighborhood of the band was fixed and finite, a situation that was termed ‘birth of a cut’ or ‘first colonization’. We now consider the transitional regime where this microscopic population in the new band grows without bounds but at a slower rate than the size of the matrix. The local population in the new band organizes in a ‘mesoscopic’ regime, in between the macroscopic behavior of the full system and the previously studied microscopic one. The mesoscopic colony may form a finite number of new bands, with a maximum number dictated by the degree of criticality of the original potential. We describe the delicate scaling limit that realizes and controls the mesoscopic colony. The method we use is the steepest descent analysis of the Riemann–Hilbert problem that is satisfied by the associated orthogonal polynomials.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction and result

The phenomenon that we want to investigate in this paper goes under the name of ‘birth of a cut’ [4, 10, 16] or ‘colonization of an outpost’ [2, 3], namely the transition when one or more new spectral bands open in the asymptotic spectrum of the model. In particular, we want to focus on the transition between the *microscopic* regime (of finite number of eigenvalues) and the macroscopic regime (where the number of eigenvalues scales like N); we call the intermediate regime the *mesoscopic* regime.

While the paper does not aim at being propaedeutic to the topic of random matrices, in this section we recall some general facts about the unitary random matrix model so as to set the context. The unitary random matrix model is defined by the probability distribution

$$Z_{n,N}^{-1} \exp\left(-\frac{N}{T} \text{Tr} V(M)\right) dM, \quad Z_{n,N} = \int_{\mathcal{H}_n} \exp\left(-\frac{N}{T} \text{Tr} V(M)\right) dM, \quad (1.1)$$

on the space \mathcal{H}_n of Hermitian $n \times n$ matrices M , with V being a real analytic function (the *potential*) that satisfies

$$\lim_{x \rightarrow \pm\infty} \frac{V(x)}{\log(x^2 + 1)} = +\infty.$$

The eigenvalues x_1, \dots, x_n of the matrices in this ensemble are distributed according to the probability distribution (see, e.g., [9, 15])

$$\mathcal{P}^{(n,N)}(x_1, \dots, x_n) d^n x = \hat{Z}_{n,N}^{-1} e^{-\frac{N}{T} \sum_{j=1}^n V(x_j)} \prod_{j < k} (x_j - x_k)^2 dx_1, \dots, dx_n, \quad (1.2)$$

where $\hat{Z}_{n,N}$ is the normalization constant.

The correlation functions of the eigenvalues are related to orthogonal polynomials (see e.g. [9, 15]): let $\{\pi_n(x)\}_{n \in \mathbb{N}}$ be the degree n monic orthogonal polynomials with weight $e^{-NV(x)}$ on \mathbb{R} [19]:

$$\int_{\mathbb{R}} \pi_n(x) \pi_m(x) e^{-\frac{N}{T} V(x)} dx = h_n \delta_{nm}. \quad (1.3)$$

Let us construct the correlation kernel by

$$K_{n,N}(x, x') = e^{-\frac{1}{2} \frac{N}{T} (V(x)+V(x'))} \sum_{j=0}^{n-1} \frac{\pi_j(x) \pi_j(x')}{h_j}.$$

By the Christoffel–Darboux formula, this kernel can be expressed in terms of the two orthogonal polynomials $\pi_n(x)$ and $\pi_{n-1}(x)$ instead of the whole sum:

$$K_{n,N}(x, x') = e^{-\frac{1}{2} \frac{N}{T} (V(x)+V(x'))} \frac{\pi_n(x) \pi_{n-1}(x') - \pi_n(x') \pi_{n-1}(x)}{h_{n-1}(x - x')}. \quad (1.4)$$

The basis of our analysis relies on the Fokas–Its–Kitaev formulation [11, 12] of OPs in terms of the following RHP for the 2×2 matrix $Y(z)$ (for brevity, we drop the explicit dependence of Y on n):

$$Y_+(x) = Y_-(x) \begin{bmatrix} 1 & e^{-\frac{N}{T} V(x)} \\ 0 & 1 \end{bmatrix}, \quad Y(z) \sim (\mathbf{1} + \mathcal{O}(z^{-1})) \begin{bmatrix} z^n & 0 \\ 0 & z^{-n} \end{bmatrix}, \quad (1.5)$$

and the polynomial $\pi_n(z)$ is simply $Y_{11}(z)$, while the kernel is recovered from

$$K_{n,N}(x, x') = e^{-\frac{1}{2} \frac{N}{T} (V(x)+V(x'))} \frac{[Y^{-1}(x)Y(x')]_{21}}{-2\pi i(x - x')}. \quad (1.6)$$

Then the m -point joint probability distribution function can be written as the determinant of the kernel (1.4) [9, 15, 17] by

$$\mathcal{R}_m^{(n,N)}(x_1, \dots, x_m) := \det(K_{n,N}(x_j, x_k))_{1 \leq j, k \leq m}.$$

In the limit $\lim_{n,N \rightarrow \infty} \frac{n}{N} = 1$, the eigenvalue density $\frac{\mathcal{R}_1^{(n,N)}(x)}{n}$ of the ensemble (1.1) is asymptotic to the *equilibrium measure* $\rho(x)$ [8, 13, 18],

$$\lim_{n,N \rightarrow \infty, \frac{n}{N} \rightarrow 1} \frac{\mathcal{R}_1^{(n,N)}(x)}{n} = \rho(x),$$

where $\rho(x) dx = d\mu_{\min}(x)$ is the normalized density of the unique measure $\mu_{\min}(x)$ that minimizes the energy

$$I(\mu) = -T \int_{\mathbb{R}} \int_{\mathbb{R}} \log|x - y| d\mu(x) d\mu(y) + \int_{\mathbb{R}} V(x) d\mu(x)$$

among all Borel probability measures μ on \mathbb{R} . The fact that $\mu_{\min}(x)$ admits a probability density follows from the assumption that $V(x)$ is real and analytic [5]. Moreover, it was shown *ibidem* that for real and analytic $V(x)$, the equilibrium measure is supported on a finite union of intervals.

1.1. *Colonization at an outpost*

The following conditions are satisfied by the equilibrium density $\rho(x)$ [8, 18]:

$$\begin{aligned} 2T \int_{\mathbb{R}} \log|x - s| \rho(s) ds - V(x) &= \ell, & x \in \text{Supp}(\rho), \\ 2T \int_{\mathbb{R}} \log|x - s| \rho(s) ds - V(x) &\leq \ell, & x \in \mathbb{R}/\text{Supp}(\rho), \end{aligned} \tag{1.7}$$

for some constant ℓ (also known as *Robin’s constant*). For a generic potential $V(x)$, the inequality in (1.7) is satisfied strictly outside the support. Suppose however that there is some point $x_0 \notin \text{Supp}(\rho(x))$ where the inequality is not strict

$$2T \int_{\mathbb{R}} \log|x_0 - s| \rho(s) ds - V(x_0) = \ell. \tag{1.8}$$

Such a potential V is called *irregular* [7]; a small perturbation of the potential may induce a new interval of support of ρ to form around x_0 . We may think of this phenomenon as the eigenvalues colonizing the point x_0 , which we will call the *outpost*. This situation has been considered previously and the term ‘birth of new cut’ was used in some of the studies [4, 10, 16].

In the studies [2–4, 10, 16], the colonization phenomenon was considered when a finite number of eigenvalues start appearing in the outpost x_0 . It was shown that the eigenvalue statistics near the outpost can be described by that of a finite-size Hermitian matrix ensemble, or a *microscopic* ensemble.

1.2. *Genus transition in random matrix models: the proliferation of a colony*

Some interesting questions about how new intervals in the support are forming remained unanswered, for example, whether several intervals in the support can form simultaneously or they have to form one after another; how to describe the eigenvalue statistics when the number of eigenvalues in the colony becomes large. This paper aims to address some of these questions, namely, we want to analyze the transition from the ‘first colonization’ to the situation where one or more new intervals are fully formed. A schematic view is shown in figure 1. Since the size of the colony, though small compared to the main cut, is taken $o(N)$ and unbounded, we use the term *mesoscopic colonization*.

The set-up is as follows: let $V(x)$ be a critical potential such that (1.8) is satisfied at a point x_0 outside of the support of the equilibrium measure and let the order of vanishing of (1.8) be $2\nu + 2$. In particular, at the outpost, we have

$$\varphi(x) = V(x) - 2T \int_{\mathbb{R}} \log|x - s| \rho(s) ds + l = C_0(x - x_0)^{2\nu+2}(1 + \mathcal{O}((x - x_0))), \quad C_0 > 0. \tag{1.9}$$

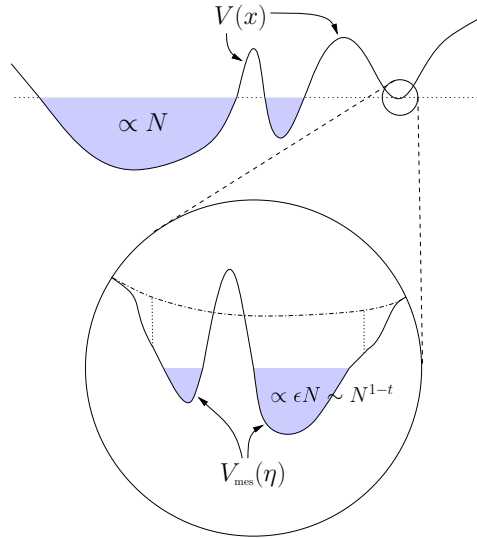


Figure 1. Mesoscopic colonization.

Without loss of generality, we will perform a translation of the problem so that $x_0 = 0$, but we will keep referring to it at x_0 not to confuse it with other zeros. The function $\varphi(x)$ is called the *effective potential* since it represents the sum of the external potential V and the Coulomb (two-dimensional) potential generated by the equilibrium distribution. Let $\mathbf{B}_J(x)$ be a bump function that is 1 inside an interval $J \subset \mathbb{R}$ around x_0 and 0 outside an interval $\tilde{J} \supset J$ around x_0 . Both J, \tilde{J} are chosen small enough so as not to contain any point of the support of ρ . We will study the perturbed model

$$Z_{n,N}^{-1} \exp\left(-\frac{N}{T} \text{Tr} \widehat{V}(M)\right) dM, \quad Z_{n,N} = \int_{\mathcal{H}_n} \exp\left(-\frac{N}{T} \text{Tr} \widehat{V}(M)\right) dM, \quad (1.10)$$

where $\widehat{V}(x)$ is a one-parameter perturbation of $V(x)$ (see figure 1)

$$\widehat{V}(x) = V(x) + \mathbf{B}_J(x) A_{\kappa,N}(x), \quad (1.11)$$

where κ is of order $\mathcal{O}(N^{1-t})$ with $0 < t < 1$, and $A(x)$ is analytic near $x = x_0$. Due to (1.9), we can define a local parameter η inside a finite neighborhood \mathbb{D} around x_0 as follows:

$$\begin{aligned} \eta &= \left(\frac{N}{\kappa T} \varphi(x)\right)^\gamma = \frac{x}{\epsilon} (1 + \mathcal{O}(x)), & (x \in \mathbb{D}), \\ \epsilon &:= \left(\frac{\kappa T}{C_0 N}\right)^\gamma, & \gamma := \frac{1}{2\nu + 2}. \end{aligned} \quad (1.12)$$

We will show that for a suitable choice of the perturbation function $A(x)$, the eigenvalues of the matrix model (1.10) are distributed on micro-cuts in \mathbb{D} whose image on the η -plane is a collection of at most $\nu + 1$ segments. In terms of the coordinate x , this support shrinks at a rate $\mathcal{O}((\kappa/N)^\gamma) = \mathcal{O}(N^{-t\gamma})$.

The bump function \mathbf{B}_J is used to keep the technicalities to a minimum and is not essential to the construction: changing between two such bump functions will introduce a difference in the description which is exponentially small (as $N \rightarrow \infty$) and hence beyond all orders of perturbation. Such a manipulation of the potential is very useful in handling the otherwise

complicated ‘double scaling limit’. The result of this paper can be encompassed in the following theorem.

Theorem 1.1. *Let V be real analytic and irregular, with an effective potential vanishing at x_0 as dictated in (1.9) and let $\kappa = \kappa_N$ be a sequence of integers⁴ such that $\kappa = \mathcal{O}(N^{1-t})$ for some $1 > t > 0$. Let $V_{\text{mes}}(s) = s^{2v+2} + \sum_{j=1}^{2v+1} t_j s^j$, $t_j \in \mathbb{R}$ be a real monic polynomial potential of degree $2v + 2$ and let $\mu_{\text{mes}}(s)$ be its normalized equilibrium measure minimizing*

$$\int_{\mathbb{R}} V_{\text{mes}}(s) d\mu_{\text{mes}}(s) - \iint_{\mathbb{R} \times \mathbb{R}} \log |s - s'| d\mu_{\text{mes}}(s) d\mu_{\text{mes}}(s'). \quad (1.13)$$

Let \mathbb{D} be a small neighborhood of x_0 in the complex plane on which η (1.12) is conformal. Then there exists a perturbation $\hat{V}(x)$ of $V(x)$, defined as in (1.11), where the function $A_{N,\kappa}(x)$ is analytic and bounded on J , of order $\mathcal{O}(\epsilon) = \mathcal{O}((\kappa/N)^\gamma) = \mathcal{O}(N^{-\gamma t})$ (and uniformly so w.r.t. N, κ and $x \in J$), such that for any u in a compact set in \mathbb{R} , we have

$$\lim_{\substack{\kappa = \mathcal{O}(N^{1-t}) \\ N \rightarrow \infty, n/N \rightarrow 1}} \frac{\epsilon}{\kappa} \mathcal{R}_1^{(n,N)} \left(x_0 + u \left(\frac{\kappa T}{C_0 N} \right)^\gamma \right) du = d\mu_{\text{mes}}(u). \quad (1.14)$$

The exact definition of $A_{N,\kappa}(x)$ is given in definition 3.1.

In particular, for a suitably chosen perturbation of the critical potential, the number of micro-cuts that are formed depends on the mesoscopic potential $V_{\text{mes}}(x)$. In particular, multiple cuts can be formed simultaneously if the equilibrium measure of $V_{\text{mes}}(x)$ is supported on multiple cuts.

The eigenvalues of (1.10), except those on the macroscopic cuts, are on the support of $d\mu_{\text{mes}}$. From (1.12), we see that the lengths of these cuts are of order $(\frac{\kappa}{N})^\gamma$. Inside these mesoscopic bands, the local eigenvalue statistics also demonstrate the usual universal behavior. We have the following.

Theorem 1.2. *Under the conditions of theorem 1.1, we have*

(i) *Let η^* be a point in the interior of the support μ_{mes} . Then we have*

$$\begin{aligned} \lim_{\substack{\kappa = \mathcal{O}(N^{1-t}) \\ N, n \rightarrow \infty}} \frac{\epsilon}{\kappa \rho_{\text{mes}}(\eta^*)} K_{n,N} \left(x_0 + \epsilon \left(\eta^* + \frac{u}{\kappa \rho_{\text{mes}}(\eta^*)} \right), x_0 + \epsilon \left(\eta^* + \frac{v}{\kappa \rho_{\text{mes}}(\eta^*)} \right) \right) \\ = \frac{\sin(\pi(u - v))}{\pi(u - v)}, \end{aligned} \quad (1.15)$$

uniformly for u, v in compact subsets of \mathbb{R} .

(ii) *Let $\text{Supp}(\rho_{\text{mes}}) = \cup_{j=1}^{2g+2} [\gamma_{2j-1}, \gamma_{2j}]$ and let $\varphi_j > 0$ be such that*

$$\rho_{\text{mes}}(z) = \frac{\varphi_j}{\pi} |z - \gamma_j|^{\frac{1}{2}} + O(z - \gamma_j),$$

as $z \rightarrow \gamma_j$, $j = 1, \dots, 2g + 2$ inside of $\text{Supp}(\mu_{\text{mes}})$. Then we have the following:

$$\begin{aligned} \lim_{\substack{\kappa = \mathcal{O}(N^{1-t}) \\ N, n \rightarrow \infty}} \frac{\epsilon}{(\kappa \varphi_j)^{\frac{2}{3}}} K_{n,N} \left(x_0 + \epsilon \left(\gamma_j + (-1)^j \frac{u}{(\kappa \varphi_j)^{\frac{2}{3}}} \right), x_0 + \epsilon \left(\gamma_j + (-1)^j \frac{v}{(\kappa \varphi_j)^{\frac{2}{3}}} \right) \right) \\ = \frac{\text{Ai}(u)\text{Ai}'(v) - \text{Ai}(v)\text{Ai}'(u)}{u - v}, \end{aligned} \quad (1.16)$$

uniformly for u, v in compact subsets of \mathbb{R} , where Ai is the Airy function.

⁴ The requirement is purely technical and could be disposed of, at the price of complicating the analysis, without changing the result.

2. Equilibrium measure in the mesoscopic problem

Consider the mesoscopic potential in theorem 1.1 with $F(\eta) := \sum_{j=1}^{2\nu+1} t_j \eta^j$,

$$V_{\text{mes}}(\eta) = \eta^{2\nu+2} + F(\eta), \quad \deg F(\eta) \leq 2\nu + 1. \quad (2.1)$$

We define as usual the corresponding g -function as the logarithmic transform of the equilibrium measure:

$$g_{\text{mes}}(\xi) = \int_{\mathbb{R}} \ln(\xi - \eta) \rho_{\text{mes}}(\eta) d\eta, \quad (2.2)$$

where $\rho_{\text{mes}}(\eta)$ is the probability measure on \mathbb{R} that minimizes the familiar energy functional

$$\mathcal{F}_{\text{mes}}[\rho] := \int V_{\text{mes}}(\eta) \rho_{\text{mes}}(\eta) d\eta + \iint \ln \frac{1}{|\eta - \xi|} \rho_{\text{mes}}(\eta) \rho_{\text{mes}}(\xi) d\eta d\xi. \quad (2.3)$$

The support of ρ_{mes} is a finite union of intervals [5], and it is possible to see that in fact it can have at most $\nu + 1$ disjoint intervals. The g -function has an expansion for large argument of the form

$$g_{\text{mes}}(\xi) = \ln \xi - \sum_{j=1}^{\infty} \frac{b_j}{\xi^j}, \quad b_j := (-1)^j \int_{\mathbb{R}} \rho_{\text{mes}}(\eta) \eta^j d\eta. \quad (2.4)$$

The mesoscopic equilibrium measure satisfies the same inequalities as in (1.7) with an appropriate (mesoscopic Robin's) constant ℓ_{mes} :

$$V_{\text{mes}}(\eta) - 2\Re g_{\text{mes}}(\eta) + \ell_{\text{mes}} = 0, \quad \eta \in \text{Supp}(\rho_{\text{mes}}), \quad (2.5)$$

$$V_{\text{mes}}(\eta) - 2\Re g_{\text{mes}}(\eta) + \ell_{\text{mes}} \geq 0, \quad \eta \notin \text{Supp}(\rho_{\text{mes}}). \quad (2.6)$$

We will need the following truncation of the expansion.

Definition 2.1. *The truncated mesoscopic g -function is defined as*

$$\widehat{g}_{\text{mes}}(\eta) = \ln \eta - \sum_{j=1}^k \frac{b_j}{\eta^j} =: \ln \eta - f_{\text{mes}}(\eta). \quad (2.7)$$

Note that we have defined both \widehat{g}_{mes} and the function f_{mes} .

The minimal level of truncation k will be determined in (3.25), but for the time being it is a parameter of our problem.

3. Singularly perturbed variational problem

In order to construct the deformation of the original problem so that we obtain the desired double-scaling limit, we need to work a bit more compared to [2, 3]. In particular, the global g -function will be modified to a certain extent because the mesoscopic colony is 'big' enough to affect the minimization problem for the macroscopic spectrum.

Let $V(x)$ be a real-analytic potential. It is known from [5] that the support of the corresponding equilibrium measure consists of a finite union of disjoint finite intervals $\sqcup[\alpha_{2j-1}, \alpha_{2j}]$. We define the complexified effective potential by the formula

$$\varphi(z) := V(z) - 2T \int_{\mathbb{R}} \rho(t) \ln(z - t) dt + \ell. \quad (3.1)$$

Due to the multivaluedness of the logarithm φ is only defined on a simply connected domain, which customarily is chosen as $\mathbb{C} \setminus (-\infty, \max \text{supp}(\rho)]$ [7]. To keep technicality to its minimal, we shall assume that x_0 lies on the right of $\text{supp} \rho$ such that φ is analytic at x_0 .

The condition that $V(x)$ is irregular (1.9) at $x_0 \notin \text{supp}(\rho)$ is translated in terms of φ as the condition $\varphi(x) = C_0 x^{2\nu+2}(1 + \mathcal{O}(x))$, $C_0 > 0$.

We will assume, for simplicity, that all other turning points are *simple*, namely at the endpoints of the intervals of the support of ρ we have $\varphi'(x) \sim C_j(x - \alpha_j)^{\frac{1}{2}}(1 + \mathcal{O}(x - \alpha_j))$ for some constant C_j 's.

The goal of this section is to define a small perturbation to the unperturbed g -function (or the unperturbed effective potential) which will serve to normalize—eventually—the RHP for the corresponding orthogonal polynomials.

Let us define the mesoscopic conformal scaling parameter η by (1.12). This definition is chosen to match the use of coordinate that was made in the previous section (section 2). Define the following Laurent polynomial in x :

$$f(x/\epsilon) := -\text{res}_{z=0} \frac{f_{\text{mes}}(\eta(z))}{z-x} dz = \sum_{j=1}^k \frac{\beta_j}{(x/\epsilon)^j}, \quad f_{\text{mes}}(\eta) \text{ as in (2.7)}. \tag{3.2}$$

We note that $\beta_j = b_j + \mathcal{O}(\epsilon)$ are analytic functions near $\epsilon = 0$.

The singularly perturbed minimization problem consists now of minimizing the following functional:

$$\begin{aligned} \mathcal{F}_\epsilon &:= \int_{\mathbb{R}} \overbrace{\left(V(x) - 2\frac{\kappa T}{N} H_\epsilon(x) \right)}{=: \tilde{V}(x)} d\mu(x) + T \iint d\mu(t) d\mu(s) \ln \frac{1}{|s-t|} \\ H_\epsilon(x) &:= \ln|x/\epsilon| - f(x/\epsilon), \\ \int d\mu(t) &= 1 - \frac{\kappa}{N}, \quad \text{supp}(\mu) \subset \mathbb{R} \setminus J, \end{aligned} \tag{3.3}$$

where the minimization is taken over the set of Borel measure that is supported on $\mathbb{R} \setminus J$. Note that, with the above definition of $H_\epsilon(x)$, the following property holds:

$$H_\epsilon(x) - \widehat{g}_{\text{mes}}(\eta) = \mathcal{O}(\epsilon) + \mathcal{O}(x)(1 + \mathcal{O}(\epsilon)). \tag{3.4}$$

We point out that the potentials $\tilde{V} = \tilde{V}(x, \epsilon)$ are admissible on $\mathbb{R} \setminus J$ in the sense of potential theory [18] for sufficiently small ϵ : let ρ_ϵ be the corresponding equilibrium measures. Then we can define the modified g -function by

$$\tilde{g}(x) := \int_{\mathbb{R}} \log(x-t) \rho_\epsilon(t) dt, \tag{3.5}$$

and the modified effective potential by

$$\tilde{\varphi}(x) := \tilde{V}(x) - 2T \tilde{g}(x) + \tilde{\ell} + T \frac{\kappa}{N} \ell_{\text{mes}}, \tag{3.6}$$

where we have written the Robin constant for the modified minimization problem as $\tilde{\ell} + T \frac{\kappa}{N} \ell_{\text{mes}}$ for convenience, and—by definition—it is such that its real part of $\tilde{\varphi}$ is zero on $\text{supp}(\rho_\epsilon)$. Note that for $\kappa = 0$ ($\epsilon = 0$), the solution of the variational problem (3.3) and the original one over the whole real axis coincide since both fulfill equations (1.7).

We can then apply the results of [14] to conclude that \tilde{V} is a regular potential (for ϵ small) on $\mathbb{R} \setminus J$. In particular, we quote the relevant.

Theorem 3.1 ([14], theorem 1.2). *Suppose V and V_m , $m = 1, 2, \dots$ are real analytic external fields on \mathbb{R} such that the following hold:*

- (i) V_m and the first three derivatives converge to V uniformly on compact subsets of \mathbb{R} ;
- (ii) the growth condition $\lim_{|x| \rightarrow \infty} V_m(x)/\ln|x| = +\infty$ holds uniformly in m .

Then the supports of the corresponding equilibrium measures are uniformly bounded. Furthermore, if V is regular then so are all V_n eventually.

We would like to point out that, although in [14], the above theorem was stated for equilibrium problems defined on \mathbb{R} , the proof in [14] is in fact valid for equilibrium measures defined in subsets of \mathbb{R} . In particular, the statements of theorem 3.1 are valid for equilibrium problems defined on the set $\mathbb{R} \setminus J$. Therefore by considering the sequence of potentials \tilde{V}_N given by

$$\tilde{V}_N(x) = V(x) - 2\frac{\kappa T}{N} H_\epsilon(x),$$

with H_ϵ defined by (3.4) and ϵ depends on N through (1.12), we see that for large enough N , the number of components in the support of the equilibrium measure ρ_ϵ is finite and constant, while its end points are smooth functions in ϵ . In fact, it is possible to derive (nonlinear) differential equations for the end points as functions of ϵ . In the appendix, we will give the result without proof, since it is not necessary to the considerations to follow.

3.1. Modified orthogonal polynomials

We choose a small interval J around the outpost that does not contain any other endpoint. We will consider the following modified orthogonality relations:

$$\int_{\mathbb{R}} p_n(x) p_m(x) e^{-\frac{N}{T} \widehat{V}(x)} dx, \tag{3.7}$$

where the perturbed potential $\widehat{V}(x)$ was given in (1.11).

For simplicity, we will also assume that $\kappa = \kappa_N$ depends on N in such a way that

- $\kappa = \kappa_N$ is an integer;
- $\kappa_N = \mathcal{O}(N^{1-t})$, $1 > t > 0$.

Were we to allow κ to be non-integer, we would have to complicate the analysis by taking into account that when κ crosses the half-integers an improved local parametrix needs to be used as in [2]. This would only lengthen (considerably) the paper while providing no further insight into the phenomenon we want to describe.

3.2. Dressing the RHP with the singularly perturbed g -function

For the orthogonal polynomials at (3.7), we take the RHP (1.5) for Y with \widehat{V} instead of V .

We define

$$\Psi(z) := e^{-\frac{N\tilde{\gamma}}{2T}\sigma_3} e^{-\frac{\kappa \ell_{\text{mes}}}{2}\sigma_3} \epsilon^{-\kappa\sigma_3} Y(z) e^{-\kappa(H_\epsilon(z) - \frac{\ell_{\text{mes}}}{2})\sigma_3} e^{-N(\tilde{g}(z) - \frac{\tilde{\gamma}}{2T})\sigma_3}. \tag{3.8}$$

The various prefactors of $Y(z)$ above are only to ensure that $\Psi(z) = \mathbf{1} + \mathcal{O}(z^{-1})$, the $\epsilon^{-\kappa\sigma_3}$ term coming to compensate the term $\ln(x/\epsilon)$ that appears in H_ϵ (3.3). In this way, the g -function is ‘stripping off’ the outer parametrix from ‘all’ the zeros including those at the outpost. This approach is different from that in [2, 3] and actually closer to [4, 16].

Let us now perform the ‘opening of lenses’ procedure to the Riemann–Hilbert problem of Ψ . Let the support of ρ_ϵ be $\text{Supp}(\rho_\epsilon) = \cup_{j=1}^{h+1} [\lambda_{2j-1}, \lambda_{2j}]$ and define the lens contours around these intervals as in figure 2. Let us now define the matrix $S(z)$ as follows:

$$S(z) := \begin{cases} \Psi(z), & z \in \mathbb{C} \setminus (L_{+j} \cup L_{-j}), \\ \Psi(z) \begin{pmatrix} 1 & 0 \\ \mp e^{\frac{N}{T} \tilde{\varphi}(z)} & 1 \end{pmatrix}, & z \in L_{\pm j}. \end{cases} \tag{3.9}$$

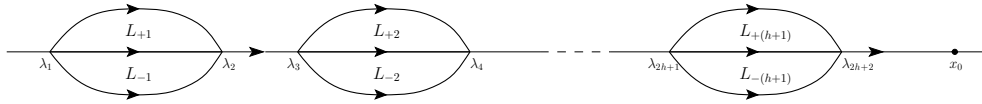


Figure 2. Contours after lens opening. The union of the lens contours is referred to as Ξ .

Let Ξ be the union of the lens contours, then $S(z)$ satisfies the following Riemann–Hilbert problem:

1. $S(z)$ is analytic in $\mathbb{C} \setminus ([\lambda_1, \lambda_{2h+2}] \cup \Xi)$,
2. $S_+(z) = S_-(z)J_S(z)$, $z \in \mathbb{R} \cup \Xi$,
3. $S(z) = I + O(z^{-1})$, $z \rightarrow \infty$,

where $J_S(z)$ is the jump matrix

$$J_S(z) := \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{\frac{N}{T}\tilde{\varphi}(z)} & 1 \end{pmatrix}, & z \in \Xi, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \text{Supp}(\rho_\epsilon), \\ \begin{pmatrix} e^{-2\pi i n \alpha_j} & e^{-\frac{N}{T}\tilde{\varphi}(z)} \\ 0 & e^{2\pi i n \alpha_j} \end{pmatrix}, & z \in (\lambda_{2j}, \lambda_{2j+1}), \end{cases} \quad (3.11)$$

where $\alpha_j = \mu_\epsilon((-\infty, \lambda_{2j}])$ are real constants. The equilibrium measure ρ_ϵ satisfies inequalities of the form (1.7) on $\mathbb{R} \setminus J$ (with V replaced by \tilde{V} and ρ replaced by ρ_ϵ). This implies that away from x_0 and the end points of $\text{Supp}(\rho_\epsilon)$, the jump off-diagonal entries of the first and last matrices in (3.11) are exponentially small in N . This suggests that the Riemann–Hilbert problem for $S(z)$ can be approximated by a ‘model Riemann–Hilbert problem’,

1. $S_+^{(\infty)}(z) = S_-^{(\infty)}(z)J_\infty(z)$, $z \in [\lambda_1, \lambda_{2h+2}]$,
2. $S^{(\infty)}(z) = I + O(z^{-1})$, $z \rightarrow \infty$,

where J_∞ is the following jump matrix:

$$J_\infty(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \bigcup_j (\lambda_{2j-1}, \lambda_{2j}), \\ \begin{pmatrix} e^{-2\pi i n \alpha_j} & 0 \\ 0 & e^{2\pi i n \alpha_j} \end{pmatrix}, & z \in (\lambda_{2j}, \lambda_{2j+1}). \end{cases}$$

Away from x_0 and the endpoints λ_j , it can then be shown that the ‘model Riemann–Hilbert problem’ (3.12) serves as a good approximation to (3.10). The Riemann–Hilbert problem (3.12) can be solved by the use of hyperelliptic theta function as in [7]. Since the exact form of the solution is not important to our analysis, we shall not recite it here but refer the readers to [7]. (For curious readers, J_∞ corresponds to $M^{(\infty)}$ in lemma 4.3 of [7], and Ψ corresponds to $M^{(1)}$ in (4.6)–(4.8) of [7].)

Near the points x_0 and λ_j , the model Riemann–Hilbert problem (3.12) becomes a poor approximation to (3.10) and a local solution $S^{(p)}(z)$ to (3.10) problem must be sought. Moreover, these local solutions (local parametrices) must match the solution $S^{(\infty)}(z)$ of (3.12) up to terms that vanishes as $N \rightarrow \infty$. In other words, we want these local parametrices to have the following properties:

1. $S_+^{(p)}(z) = S_-^{(p)}(z)J_S(z)$, $z \in (\mathbb{R} \cup \Xi)$, inside a small neighborhood of x_0 or λ_j ;
2. $S^{(p)}(z) = (I + o(1))S^{(\infty)}(z)$, z in the boundary of these neighborhoods.

The local parametrices near the points λ_j can be constructed using Airy functions as was done in [7] (see (4.76), (4.92) and (4.102) in [7]). Again, we shall not recite these formulae here as the exact form of these formulae is not important to our analysis.

3.3. Local parametrix at the mesoscopic colony

We shall now consider the local parametrix near the point x_0 . The jumps on J for $S(z)$ are given by

$$S_+(x) = S_-(x) \begin{bmatrix} 1 & e^{-\frac{N}{T}(\widehat{V}-2T\widetilde{g}+\widetilde{\ell})+\kappa(2H_\epsilon(x)-\ell_{\text{mes}})} \\ 0 & 1 \end{bmatrix}. \tag{3.14}$$

In order to have locally the (simplest form of) RHP for the mesoscopic potential, we need to have

$$\begin{aligned} & \frac{N}{T}(\widehat{V}(x) - 2T\widetilde{g}(x) + \widetilde{\ell}) - 2\kappa H_\epsilon(x) + \kappa \ell_{\text{mes}} \stackrel{\text{set}}{=} \kappa (V_{\text{mes}}(\eta) - 2\widehat{g}_{\text{mes}}(\eta) + \ell_{\text{mes}}) \\ & = \underbrace{\frac{N}{T}(V(x) - 2Tg(x) + \ell) + \kappa F(\eta) - 2\kappa \widehat{g}_{\text{mes}}(\eta) + \kappa \ell_{\text{mes}}}_{=\kappa \eta^{2\nu+2}}. \end{aligned} \tag{3.15}$$

Simplifying the above expression, we have

Definition 3.1. The perturbation function $A(z) := A_{\kappa,N}(z)$ to be used in the perturbed potential (1.11) is defined as

$$A(z) = \widehat{V} - V = -(2T(g - \widetilde{g}) + (\ell - \widetilde{\ell})) + \frac{\kappa T}{N}(F(\eta) + 2(H_\epsilon(x) - \widehat{g}_{\text{mes}}(\eta))). \tag{3.16}$$

Remark 3.1. We recall what are the input data in definition 3.1, so as to make clear that the definition is not circular: we need

- the unperturbed non-regular potential $V(z)$ (with the property (1.9));
- the mesoscopic potential $V_{\text{mes}}(\eta) = \eta^{2\nu+2} + F(\eta)$, with $F(\eta)$ being an arbitrarily chosen polynomial of degree $2\nu + 1$;
- the order of truncation k .

The other functions appearing in (3.16) are H_ϵ (defined in (3.3)), the truncated g -function (definition 2.1).

In order to manifest the analytic properties of $A(x) = \widehat{V} - V$, we point out that the singularities of $H_\epsilon(z)$ and \widehat{g}_{mes} cancel out precisely by (3.4) to give a locally analytic function in the neighborhood of the outpost. Also the largest deviation is given by the term $\frac{\kappa T}{N}F(\eta) = \epsilon^{2\nu+2}\mathcal{O}(\epsilon^{-2\nu-1}) = \mathcal{O}(\epsilon)$. The deviation $2T(g - \widetilde{g}) + (\ell - \widetilde{\ell})$ is bounded by $\mathcal{O}(\epsilon)$ because the endpoints of $\text{supp } \rho_\epsilon$ are a smooth function in ϵ , as remarked after theorem 3.1 (see, for instance, the lemma Appendix A.1).

With this position for A we have the following RHP in the neighborhood of the outpost:

$$\begin{aligned} S_+ &= S_- \begin{bmatrix} 1 & e^{-\kappa(V_{\text{mes}}(\eta)-2\widehat{g}_{\text{mes}}(\eta)+\ell_{\text{mes}})} \\ 0 & 1 \end{bmatrix}, \\ S(z) &= \mathcal{O}(z^0) e^{-\kappa H_\epsilon(z)\sigma_3} = \mathcal{O}(\eta^0) e^{-\kappa \widehat{g}_{\text{mes}}(\eta)\sigma_3}, \quad \eta \rightarrow 0. \end{aligned} \tag{3.17}$$

The growth behavior at the origin is obtained from the definition of Ψ (3.8) and S (3.9). The reason why the two behaviors at $z = 0$ on the second line of (3.17) are equivalent is due to the fact that $H_\epsilon(z)$ is precisely the singular part of \widehat{g}_{mes} , as follows from (3.3) and (3.2).

Let $P_j(\eta) = P_j(\eta; \kappa)$ be the *monic* orthogonal polynomials for the (varying) measure $e^{-\kappa V_{\text{mes}}(\eta)} d\eta$. We want to construct an exact solution $R_\kappa(\eta)$ to the jump condition (3.17) such that on $\partial\mathbb{D}$ it behaves as $\mathbf{1} + \mathcal{O}(N^{-\alpha})$ for some positive α and uniformly in η .

Consider the matrix

$$R_\kappa(\eta) = e^{-\frac{1}{2}\kappa\ell_{\text{mes}}\sigma_3} \begin{bmatrix} P_\kappa(\eta) & \mathcal{C}[P_\kappa](\eta) \\ \frac{-2i\pi}{h_{\kappa-1}} P_{\kappa-1}(\eta) & \frac{-2i\pi}{h_{\kappa-1}} \mathcal{C}[P_{\kappa-1}](\eta) \end{bmatrix} e^{-\kappa\widehat{g}_{\text{mes}}\sigma_3} e^{\frac{1}{2}\kappa\ell_{\text{mes}}\sigma_3}. \quad (3.18)$$

It is immediately possible to verify that it solves the following jump condition and asymptotic behavior:

$$R_\kappa(\eta)_+ = R_\kappa(\eta)_- \begin{bmatrix} 1 & e^{-\kappa(V_{\text{mes}}(\eta) - 2\widehat{g}_{\text{mes}}(\eta) + \ell_{\text{mes}})} \\ 0 & 1 \end{bmatrix}, \quad (3.19)$$

$$R_\kappa(\eta) = \mathbf{1} + \mathcal{O}\left(\frac{1}{\eta}\right) \quad \eta \rightarrow \infty. \quad (3.20)$$

$$R_\kappa(\eta) = \mathcal{O}(1) e^{-\kappa\widehat{g}_{\text{mes}}\sigma_3}, \quad \eta \rightarrow 0. \quad (3.21)$$

The last error term $\mathcal{O}(\eta^{-1})$ in (3.20) depends on κ and we need to control this uniformly in the limit $\kappa \rightarrow \infty$. By performing the lens opening procedure in (3.10) to the Riemann–Hilbert problem of $R_\kappa(\eta)$ and making use of the properties (2.5) of g_{mes} , we can again approximate this Riemann–Hilbert problem by the solution $S_{\text{mes}}^{(\infty)}(\eta)$ of a model Riemann–Hilbert problem of the form (3.12). Therefore, as $\kappa \rightarrow \infty$ we can obtain expressions

$$R_\kappa(\eta) = \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{\kappa E_{\text{mes}}}\right) \right) S_{\text{mes}}^{(\infty)}(\eta) e^{\kappa g_{\text{mes}}(\eta)\sigma_3} e^{-\kappa\widehat{g}_{\text{mes}}(\eta)\sigma_3}, \quad (3.22)$$

away from the support of μ_{mes} . Note that mesoscopic Robin’s constant disappears by virtue of our well-crafted choice of perturbation.

The first factor in (3.22) comes from the error of the mesoscopic error matrix, and E_{mes} is determined by the nature of the mesoscopic system; for a usual situation with regular mesoscopic potential, we have $E_{\text{mes}} = 1$. (To learn how this error term might occur, one may look at theorem 7.171 and equation (7.73) in [8].)

From (3.12), we see that

$$S_{\text{mes}}^{(\infty)}(\eta) = \mathbf{1} + \mathcal{O}(\eta^{-1}), \quad \eta \rightarrow \infty,$$

where the error term is also bounded in κ as $\kappa \rightarrow \infty$.

The trailing exponential factors in (3.22) determine the minimal order of the truncation (2.7): using definition 2.1, we have $g_{\text{mes}}(\eta) - \widehat{g}_{\text{mes}}(\eta) = \mathcal{O}(1/\eta^{k+1})$. Therefore, for η on the boundary $\partial\mathbb{D}$ we have the uniform estimate

$$R_\kappa(\eta) = \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{\kappa E_{\text{mes}}}\right) \right) \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{\eta}\right) \right) \left(\mathbf{1} + \mathcal{O}\left(\frac{\kappa}{\eta^{k+1}}\right) \right) \\ = \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{\kappa E_{\text{mes}}}\right) \right) \left(\mathbf{1} + \mathcal{O}\left(\frac{\kappa^\gamma}{N^\gamma}\right) \right) \left(\mathbf{1} + \mathcal{O}\left(\frac{\kappa^{\gamma(k+1)+1}}{N^{\gamma(k+1)}}\right) \right). \quad (3.23)$$

★

The last contribution to the error term marked with ★ in (3.23) is the most important one: demanding that the error decay imposes a condition on the minimal k of the truncation in definition (2.1). Indeed, the growth of κ must be such that there exists a minimal k_{min} for which the last term is $o(1)$. In other words, the order of growth of the colony must be

$$\kappa < N^{1 - \frac{1}{\gamma(k+1)+1}}, \quad (3.24)$$

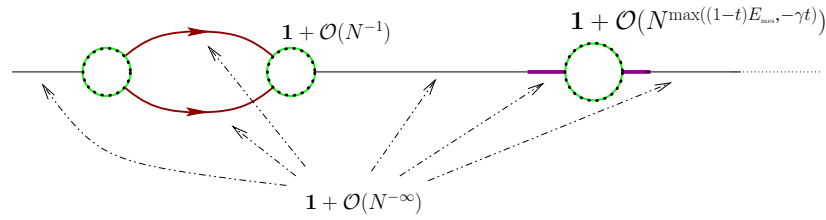


Figure 3. The order of the jumps of the typical residual Riemann–Hilbert problem for the error term, here depicted for a one-cut situation with regular endpoints (where the Airy local parametrix can be used).

for k sufficiently large.

If $\kappa = \mathcal{O}(N^{1-t})$ for some $0 < t < 1$ then we need to choose k so that

$$\frac{1}{\gamma(k+1)+1} < t \Leftrightarrow k > (2\nu+2)\left(\frac{1}{t}-1\right)-1. \tag{3.25}$$

This determines the minimal order of truncation in (2.1) and in all the analysis that followed. The error bound is then dominated by one of the terms in (3.23). Choosing the next-to-minimal k —which we do henceforth—yields an error bound $1 + \mathcal{O}(N^{-\gamma t})$ and this already matches the error coming from the middle term in (3.23) (which cannot be improved more along these lines). Thus, in general, we will have an overall error term

$$\mathbf{1} + \mathcal{O}(N^{\max((t-1)E_{\text{mes}}, -\gamma t)}). \tag{3.26}$$

4. Final transformation of the Riemann–Hilbert problem

Let us now carry out the error analysis and show that the matrix $S(z)$ in (3.10) can indeed be approximated by $R_\kappa(\eta)$ at the outpost.

Let $S^{(j)}(z)$ be the local parametrices near the points λ_j constructed from Airy functions, and let us define the matrix $S^{\text{as}}(z)$ by

$$S^{\text{as}}(z) = \begin{cases} S^{(\infty)}(z) & \text{outside the local regions} \\ S^{(\infty)}(z)S^{(j)}(z) & \text{near the endpoint } \lambda_j \\ S^{(\infty)}(z)R_\kappa(z) & \text{at the outpost.} \end{cases} \tag{4.1}$$

The error matrix $\mathcal{E} := S^{\text{as}}S^{-1}$ has several residual jumps on the contours shown in figure 3. In particular, on the boundary of the disk at the outpost we have

$$\mathcal{E}_+(\mathcal{E}_-)^{-1} = S^{\text{as}}R_\kappa(S^{\text{as}})^{-1} = \mathbf{1} + \mathcal{O}(N^{-\Delta}), \tag{4.2}$$

$$\Delta := \max((1-t)E_{\text{mes}}, -\gamma t). \tag{4.3}$$

Then, using well-established techniques (see, e.g., [6] section 7) we obtain

$$\mathcal{E}(z) = I + \mathcal{O}\left(\frac{1}{N^\Delta(|z|+1)}\right), \tag{4.4}$$

uniformly in \mathbb{C} . This shows that the matrix $S(z)$ can indeed be approximated by $R_\kappa(\eta)$ near x_0 .

It should be clear that the error term of the analysis becomes $\mathcal{O}(1)$ as $t \rightarrow 0_+$, namely, as κ grows at the same order as N (at which point the new gaps must be ‘fully formed’): more

and more terms need to be added to the truncation 2.1. Eventually one must solve an exact minimization problem when the colony is fully grown and the transition from mesoscopic to macroscopic will be complete.

5. Proof of theorem 1.1 and theorem 1.2

Retracing the transformations we obtain the asymptotic form of our orthogonal polynomial near the outpost (which we are interested in) by

$$\pi_n(z) = [Y(z)]_{11} = \left[e^{\frac{N\tilde{\ell}}{2T}\sigma_3} e^{\frac{\kappa\ell_{\text{mes}}}{2}\sigma_3} \epsilon^{\kappa\sigma_3} (\mathbf{1} + \mathcal{O}(N^{-\Delta})) S^{\text{as}} e^{\kappa(H_\epsilon(z) - \frac{\ell_{\text{mes}}}{2})\sigma_3} e^{N(\tilde{g}(z) - \frac{\tilde{\ell}}{2T})\sigma_3} \right]_{11} \quad (5.1)$$

$$= e^{\kappa \log \epsilon + \kappa H_\epsilon(z) + N\tilde{g}(z)} [(\mathbf{1} + \mathcal{O}(N^{-\Delta})) S^{(\infty)}(z) R_\kappa(\eta)]_{11} \quad (5.2)$$

$$= e^{\kappa \log \epsilon + \kappa H_\epsilon(z) + N\tilde{g}(z)} \left[(\mathbf{1} + \mathcal{O}(N^{-\Delta})) S^{(\infty)}(z) \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{\kappa E_{\text{mes}}}\right) \right) \times S_{\text{mes}}^{(\infty)}(\eta) e^{\kappa g_{\text{mes}}(\eta)\sigma_3} e^{-\kappa \widehat{g}_{\text{mes}}(\eta)\sigma_3} \right]_{11} \quad (5.3)$$

$$= e^{\kappa \log \epsilon + \kappa H_\epsilon(z) + N\tilde{g}(z) + \kappa g_{\text{mes}}(\eta) - \kappa \widehat{g}_{\text{mes}}(\eta)} \times \left[(\mathbf{1} + \mathcal{O}(N^{-\Delta})) S^{(\infty)}(z) \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{\kappa E_{\text{mes}}}\right) \right) S_{\text{mes}}^{(\infty)}(\eta) \right]_{11} \quad (5.4)$$

$$= (1 + \mathcal{O}(N^{-\Delta})) e^{N\tilde{g}(z) + \kappa g_{\text{mes}}(\eta) + \kappa \log \epsilon} [S^{(\infty)}(0) S_{\text{mes}}^{(\infty)}(\eta)]_{11}, \quad (5.5)$$

where Δ was introduced in equation (4.3). In the third to last equality, we have used (3.22) and, in the last equality, we have used (3.4). The result indicates that $\pi_n(z)$ has N roots that are captured by $\tilde{g}(z)$ away from the outpost and κ roots that are captured by $g_{\text{mes}}(\eta)$ near the outpost; the factor $e^{\kappa \log \epsilon}$ arises from the scaling between η and z .

To prove theorem 1.1, we need to show that $\frac{\epsilon K_{n,N}(x,x)}{\kappa}$ is equal to ρ_{mes} . First, we write the asymptotics of Y near the outpost as follows:

$$Y(z) = e^{\frac{N\tilde{\ell}}{2T}\sigma_3} e^{\frac{\kappa\ell_{\text{mes}}}{2}\sigma_3} \epsilon^{\kappa\sigma_3} (\mathbf{1} + \mathcal{O}(N^{-\Delta})) S^{(\infty)}(z) \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{\kappa E_{\text{mes}}}\right) \right) S_{\text{mes}}^{(\infty)}(\eta) \quad (5.6)$$

$$\times \exp\left(\kappa g_{\text{mes}}(\eta) - \kappa \widehat{g}_{\text{mes}}(\eta) + \kappa \left(H_\epsilon(z) - \frac{\ell_{\text{mes}}}{2}\right) + N \left(\tilde{g}(z) - \frac{\tilde{\ell}}{2T}\right)\right) \sigma_3 \quad (5.7)$$

$$= e^{\frac{N\tilde{\ell}}{2T}\sigma_3} e^{\frac{\kappa\ell_{\text{mes}}}{2}\sigma_3} \epsilon^{\kappa\sigma_3} (\mathbf{1} + \mathcal{O}(N^{-\Delta})) S^{(\infty)}(z) \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{\kappa E_{\text{mes}}}\right) \right) S_{\text{mes}}^{(\infty)}(\eta) \quad (5.8)$$

$$\times \exp\left(-\frac{\kappa}{2}\phi_{\text{mes}} + \frac{N}{2T}\widehat{V}(z)\right) \sigma_3, \quad \phi_{\text{mes}} := V_{\text{mes}} - 2g_{\text{mes}} + \ell_{\text{mes}}. \quad (5.9)$$

This is the asymptotic form *away* from the mesoscopic cut. On the mesoscopic cut (approached from the upper half-plane, for instance), one needs to right-multiply the above Y by $\begin{bmatrix} \exp\frac{1}{\gamma}\widehat{V}(z) & 0 \\ 1 & 1 \end{bmatrix}$, to take into account the lens opening around the mesoscopic cut. Now we can evaluate

$$\lim_{n,\kappa \rightarrow \infty} \frac{\epsilon K_{n,N}(x, x)}{\kappa} = \lim_{n,\kappa \rightarrow \infty} \frac{\epsilon e^{-\frac{N}{T} \widehat{V}(x)}}{2\pi i \kappa} [Y^{-1}(x)Y'(x)]_{21} \tag{5.10}$$

$$= \frac{\epsilon e^{-\frac{N}{T} \widehat{V}(x)}}{2\pi i \kappa} \left[\begin{array}{cc} 1 & 0 \\ -\exp \frac{N}{T} \widehat{V}(x) & 1 \end{array} \right] \left(-\frac{\kappa}{2\epsilon} \frac{d\phi_{\text{mes}}}{d\eta} \sigma_3 \right) \left[\begin{array}{cc} 1 & 0 \\ \exp \frac{N}{T} \widehat{V}(x) & 1 \end{array} \right]_{21} \tag{5.11}$$

$$= \frac{1}{2\pi i} \frac{d\phi_{\text{mes}}}{d\eta}, \tag{5.12}$$

which is exactly ρ_{mes} (if the cut is approached from the upper half-plane) by the definition of the mesoscopic g function (2.2) and ϕ_{mes} (5.9). This proves theorem 1.1.

Let us also show the sine kernel at the bulk of mesoscopic spectrum. Let $x = x_0 + \epsilon \left(\eta^* + \frac{u}{\kappa \rho_{\text{mes}}(\eta^*)} \right)$ and $x' = x_0 + \epsilon \left(\eta^* + \frac{v}{\kappa \rho_{\text{mes}}(\eta^*)} \right)$. Then we have

$$\lim_{n,\kappa \rightarrow \infty} \frac{\epsilon K_{n,N}(x, x')}{\kappa \rho_{\text{mes}}(\eta^*)} = \lim_{n,\kappa \rightarrow \infty} \epsilon e^{-\frac{1}{2} \frac{N}{T} (V(x)+V(x'))} \frac{[Y^{-1}(x)Y(x')]_{21}}{-\kappa \rho_{\text{mes}}(\eta^*) 2\pi i (x - x')} \tag{5.13}$$

$$= \lim_{n,\kappa \rightarrow \infty} \frac{\epsilon}{\kappa \rho_{\text{mes}}(\eta^*)} \frac{e^{-\frac{1}{2} \frac{N}{T} (V(x)+V(x'))}}{-2\pi i (x - x')} \tag{5.14}$$

$$\begin{aligned} & \times \left[\begin{array}{cc} 1 & 0 \\ -\exp \frac{N}{T} \widehat{V}(x) & 1 \end{array} \right] e^{-\frac{N}{2T} \widehat{V}(x) \sigma_3} e^{\frac{\kappa}{2} (\phi_{\text{mes}}(\eta) - \phi_{\text{mes}}(\eta')) \sigma_3} e^{\frac{N}{2T} \widehat{V}(x') \sigma_3} \\ & \times \left[\begin{array}{cc} 1 & 0 \\ \exp \frac{N}{T} \widehat{V}(x') & 1 \end{array} \right]_{21} \end{aligned} \tag{5.15}$$

$$= \lim_{n,\kappa \rightarrow \infty} \frac{\epsilon 2i \sin \frac{\kappa}{2} (\phi_{\text{mes}}(\eta) - \phi_{\text{mes}}(\eta'))}{\kappa \rho_{\text{mes}}(\eta^*) 2\pi i (x - x')} = \frac{\sin \pi (u - v)}{\pi (u - v)}, \tag{5.16}$$

which proves (1.15) of theorem 1.2.

From the asymptotics of Y , it is possible to prove (1.16) in the universality theorem 1.2. Since the computation is identical to the standard case (in chapter 8 of [8]) and is similar to the proof of (1.15), we will not pursue the calculation here.

We now comment on why the choice of bump function is totally irrelevant; indeed, on the real axis and outside of \mathbb{D} the jump is exponentially close to the identity matrix (uniformly). Changing bump function trades such a jump by another one, equally close to the identity jump, while leaving the jumps identical within \mathbb{D} . As a consequence, the ratio of the solutions corresponding to different choices of bump functions would solve a RHP with jumps exponentially close to the identity everywhere (and in $L^\infty \cap L^2$). Thus, the two solutions would differ only by exponentially suppressed terms, well beyond any order of perturbation.

5.1. Universality of the behavior

The perturbation of the potential has been chosen in the contrived form (1.11, definition 3.1) to eventually yield the simplest form for the local Riemann–Hilbert problem in section 3.3; the gist of all the construction is such that in the scaling coordinate η the jump on the interval J is given precisely by (3.19). As often happens [2, 3], the logic of our construction is slightly backward from the more conventional approaches [4, 10, 16]: we guess what local RHP would give the phenomenon we expect on heuristic grounds, and try to ‘reverse-engineer’ the appropriate deformation of the potential. This approach, while completely rigorous and also simpler to implement, is possibly not the most transparent to the reader. The perturbation

$A(z)$ (3.1) is (a) analytic in z and (b) of order $(\kappa/N)^{\frac{1}{2\nu+2}}$: from a heuristic point of view (based also on similar set-ups in the study of the universal unfolding of singularities [1]), it is natural to expect that these are the only relevant features to generate a mesoscopic colonization. Of course there is much more detailed information that goes into our approach, because the mesoscopic colonization as we described—with a fixed (*i.e.* non-scaling in N) local matrix model—can only be obtained as a multi-scaling limit; isolating κ eigenvalues requires one scaling, forming a specific local cut structure will require rather complicated scalings. While we could not find a simpler, more direct path that starts from the perturbation and ends at a full description of the scaling regime, we do not expect that such a description, while logically more appealing, would be any simpler.

6. Conclusion and generalizations

- A quite parallel analysis could be performed in the case of the colonization of a hard-edge as in [3]. While the logic is identical, there are sufficient small details that would require a separate analysis, but with the final picture being completely analogous: in that case one can also have—depending on the degree of irregularity of the unperturbed potential—a mesoscopic growth of several meso-intervals for the equilibrium measure. We believe that the analysis is not sufficiently different to require a separate paper and yet not similar enough to put it here at the expense of clarity and conciseness.
- It was also pointed out to one of us⁵ that the technique of analyzing the colonization (microscopic and mesoscopic) can be applied almost verbatim to the study of the trailing-edge of the solution of the Korteweg–deVries equation after the time of gradient catastrophe; we may to come back to this issue in a subsequent publication.
- Since the mesoscopic potential can be an arbitrary polynomial, we could choose V_{mes} as a non-regular potential such that it has a point outside the support of ρ_{mes} where the variational inequalities (1.7) fail. Thus, one may have the whole picture of microscopic/mesoscopic colonization within the analysis of the mesoscopic parametrix. By perturbing V_{mes} accordingly one could study a *multiscale colonization*. Since a polynomial potential of degree $2d + 2$ can have such a non-regular point with order at most $2d$, we can ‘embed’ the micro/mesoscopic pictures one into another at several nested scales at most 2ν times, if the macroscopic potential has an irregular point as that studied in this paper. We could call this multiscale situation the ‘Matryoshka’⁶ colonization’.

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Appendix A. Differential equations for the endpoints

Here we simply state a result (proposition Appendix A.1) that can be proved along the lines of the Buyarov–Rakhmanov equation. We use the same notation as in the text (3.6) and we set

$$y = \frac{1}{2} \tilde{V}'(x) - T \int \frac{\rho_\epsilon(s) ds}{s - x} = \tilde{\phi}'(x). \quad (\text{A.1})$$

⁵ We thank B Dubrovin and T Grava for the indication.

⁶ A Matryoshka doll is Russian toy consisting in a set of dolls of decreasing sizes placed one inside the other.

It is known that y solves a (pseudo) algebraic equation of the form

$$y^2 = F_\epsilon(x)^2 \prod_{j=1}^{2g+2} (x - \alpha_j(\epsilon)) \tag{A.2}$$

and $F_\epsilon(x)$ is a real-analytic function (depending on ϵ) with a pole of degree k at $x = 0$.

Lemma Appendix A.1. *For small non-negative ϵ , we have that (the dot means differentiation in ϵ)*

$$\omega(x) dx := \dot{y} dx \tag{A.3}$$

is the unique meromorphic differential (whose existence follows from standard arguments in algebraic geometry) on the hyperelliptic Riemann surface branched at the endpoints $w^2 = \prod_{j=1}^{2g+2} (x - \alpha_j)$ with the properties that

- (i) all periods are purely imaginary;
- (ii) $\omega(x) dx$ has poles only at the point above $x = 0$ with residues $\mp 2T$ (respectively, on each sheet);
- (iii) at $x = 0$ (on the physical sheet) it behaves as

$$\omega(x) \sim -\frac{2T}{x} + \frac{\gamma T}{C_0 \epsilon^{2\nu+1}} \partial_\epsilon \partial_x (f(x/\epsilon)). \tag{A.4}$$

Note that the second part contains poles of order strictly higher than 1 and hence corresponds to a second-kind differential.

In particular, we note that $\omega(x)$ can be written as

$$\omega(x) = \frac{R(x)}{\sqrt{\prod_{j=1}^{2g+2} (x - \alpha_j)}} \tag{A.5}$$

with $R(x)$ being a rational function of the form

$$R(x) = P_{k+1} \left(\frac{1}{x} \right) + P_{g-1}(x), \tag{A.6}$$

and $P_m(Z)$ denotes some polynomial of degree m of the indeterminate Z . The above three facts completely determine $R(x)$ as a function of α_j 's, β_j 's and ϵ .

Proposition Appendix A.1. *The endpoints solve the following differential equation:*

$$\dot{\alpha}_j = \frac{R(\alpha_j)}{F(\alpha_j) \prod_{k \neq j} (\alpha_j - \alpha_k)}, \tag{A.7}$$

$$\dot{F}_\epsilon(x) = \frac{R(x) - F_\epsilon(x) \sum_j \dot{\alpha}_j \prod_{k \neq j} (x - \alpha_k)}{\prod_j (x - \alpha_j)}. \tag{A.8}$$

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